Strategy for Testing Series: Solutions

1. Since $(-5)^{-n} = (-1/5)^n$, this is a geometric series. Because $|-1/5| < 1$, it converges.

2. Since $n^2 < n^2 + 6n = n(n+6)$ for all $n \geq 0$, we have

$$\frac{1}{n(n+6)} < \frac{1}{n^2}\,.$$

Because $\sum 1/n^2$ converges (it's a p-series with $p = 2 > 1$), the comparison test implies that $\sum 1/(n(n+6))$ also converges.

3. Clearly, the sequence $a_n = 1/(50n)$ is decreasing and converges to 0. Thus, the alternating series test implies that $\sum(-1)^{n+1}a_n$ converges.

4. Using l'Hôpital's rule, we have

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\ln n} = \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{2} = \infty\,.$$

Hence, the $\lim_{n\to\infty}(-1)^n \frac{\sqrt{n}}{\ln n}$ doesn't exist and therefore the series $\sum(-1)^{n+1}\frac{\sqrt{n}}{\ln n}$ diverges.

5. Applying the ratio test, we have

$$\lim_{n\to\infty} \frac{\frac{r^{n+1}}{(n+1)^r}}{\frac{r^n}{n^r}} = \lim_{n\to\infty} \frac{rn^r}{(n+1)^r} = \lim_{n\to\infty} r\left(\frac{n}{n+1}\right)^r = r < 1\,,$$

and therefore the series converges.

6. If $f(n) = (n^2 - n)^{-1/2}$, then $f'(n) = -\frac{1}{2}(n^2-n)^{-3/2}(2n-1)$. Thus, when $n > 1$, $f'(n) < 0$ and $f(n)$ is decreasing. Moreover, we have

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \frac{1}{\sqrt{n(n-1)}} = 0\,.$$

Therefore, the alternating series test implies $\sum 1/\sqrt{n(n-1)}$ converges.

7. Since $n^3 < n^3 + 2$, we have

$$\frac{\sqrt{n}\ln(n)}{n^3+2} < \frac{\sqrt{n}\ln(n)}{n^3}\,.$$

Furthermore, because $\ln(n) < n$ for all $n > 0$ [1], we obtain

$$\frac{\sqrt{n}\ln(n)}{n^3+2} < \frac{\sqrt{n}n}{n^3} = n^{-3/2}\,.$$

Now, the series $\sum n^{-3/2}$ converges (it's a p-series with $p = 3/2 > 1$) and hence the comparison test implies that the series $\sum \frac{\sqrt{n}\ln(n)}{n^3+2}$ also converges.

[1] If $f(x) = \frac{\ln x}{x}$, then $f'(x) = \frac{1-\ln x}{x^2}$. It follows that $f(x)$ has a unique critcal point at $x = e$. Since $f'(x) > 0$ when $x < e$ and $f'(x) < 0$ when $x > e$, the first derivative test implies that $f(x)$ has a global maximum at e. Hence, $\frac{\ln x}{x} = f(x) \leq f(e) = \frac{\ln e}{e} < 1$ which yields $\ln x < x$ for all $x > 0$.

8. Since e^x is a strictly increasing function, $e^{1/n} \leq e$ for all $n \geq 1$. Hence, we have

$$\frac{e^{1/n}}{n^{3/2}} \leq \frac{e}{n^{3/2}}.$$

Since $\sum e n^{-3/2}$ converges (it's a p-series with $p = 3/2 > 1$), the comparison test implies that $\sum e^{1/n} n^{-3/2}$ also converges.

9. If $f(n) = \frac{(n+2)(n+3)}{(n+1)^3}$ then

$$f'(n) = \frac{(2n+5)(n+1)^3 - 3(n^2+5n+6)(n+1)^2}{(n+1)^6} = -\frac{n^2+8n+13}{(n+1)^4}.$$

When $n \geq 0$, $f'(n) < 0$ and $f(n)$ is decreasing. Moreover,

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \frac{(n+1)(n+2)}{(n+1)^3} = \lim_{n\to\infty} \frac{n^2}{n^3} = 0.$$

Therefore, the alternating series test implies $\sum(-1)^n \frac{(n+1)(n+2)}{(n+1)^3}$ converges.

10. Since

$$\frac{n^n}{n!} = \left(\frac{n}{n}\right)\left(\frac{n}{n-1}\right)\left(\frac{n}{n-2}\right)\cdots\left(\frac{n}{2}\right)\left(\frac{n}{1}\right) > 1,$$

the limit $\lim_{n\to\infty} \frac{(-1)^n n^n}{n!}$ is not zero and hence the series $\sum \frac{(-1)^n n^n}{n!}$ diverges.

11. For all $n > 2$, we have

$$\frac{n!}{n^n} = \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{2}{n}\right)\left(\frac{1}{n}\right) < \left(\frac{2}{n}\right)\left(\frac{1}{n}\right) = \frac{2}{n^2}.$$

Since $\sum 2n^{-2}$ converges (it's a p-series with $p = 2 > 1$), the comparison test implies that $\sum \frac{n!}{n^n}$ converges. Finally, the absolute convergence test implies that the series $\sum(-1)^n \frac{n!}{n^n}$ also converges.

12. Applying the ratio test, we have

$$\lim_{n\to\infty} \frac{\frac{e^{n+1}}{(n+1)!}}{\frac{e^n}{n!}} = \lim_{n\to\infty} \frac{e}{n+1} = 0 < 1,$$

and hence the series $\sum \frac{e^n}{n!}$ converges.

13. Applying the ratio test, we have

$$\lim_{n\to\infty} \frac{\frac{(n+1)^2(n+1)!}{(2n+2)!}}{\frac{n^2 n!}{(2n)!}} = \lim_{n\to\infty} \frac{(n+1)^2(n+1)}{(2n+2)(2n+1)n^2} = \lim_{n\to\infty} \frac{n^3}{n^4} = 0 < 1,$$

and therefore the series $\sum \frac{n^2 n!}{(2n)!}$ converges.

14. Since $n! < n! + 2$, we have
$$\frac{3^n}{n!+2} < \frac{3^n}{n!}\,.$$
Now, applying the ratio test to the series $\sum \frac{3^n}{n!}$, we obtain
$$\lim_{n\to\infty} \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1, .$$
Thus, the series $\sum \frac{3^n}{n!}$ converges and the comparison test implies that $\sum \frac{3^n}{n!+2}$ also converges.

15. Applying the ratio test, we have
$$\lim_{n\to\infty} \frac{\frac{n+1}{(\ln(n+1))^{n+1}}}{\frac{n}{(\ln n)^n}} = \lim_{n\to\infty} \frac{n+1}{n\ln(n+1)} \cdot \left(\frac{\ln n}{\ln(n+1)}\right)^n .$$
Since $\ln x$ is a strictly increasing function, $\ln n < \ln(n+1)$ and $\frac{\ln n}{\ln(n+1)} < 1$. Hence,
$$0 \le \lim_{n\to\infty} \frac{n+1}{n\ln(n+1)} \cdot \left(\frac{\ln n}{\ln(n+1)}\right)^n \le \lim_{n\to\infty} \frac{n+1}{n\ln(n+1)} = \lim_{n\to\infty} \frac{1}{\ln(n+1) + \frac{n}{n+1}} = 0\,.$$
Therefore, $\lim_{n\to\infty} \frac{n+1}{n\ln(n+1)} \cdot \left(\frac{\ln n}{\ln(n+1)}\right)^n = 0 < 1$ and the series $\sum \frac{n}{(\ln n)^n}$ converges.

16. Applying the ratio test, we have
$$\lim_{n\to\infty} \frac{\frac{(n+1)^6 5^{n+1}}{(n+2)!}}{\frac{n^6 5^n}{(n+1)!}} = \lim_{n\to\infty} \frac{5(n+1)^6}{n^6(n+2)} = \lim_{n\to\infty} \frac{5n^6}{n^7} = 0 < 1\,,$$
and hence the series $\sum \frac{n^6 5^n}{(n+1)!}$ converges.

17. Applying the ratio test, we have
$$\lim_{n\to\infty} \frac{\frac{e^{n+1}}{(\ln(n+1))^{n+1}}}{\frac{e^n}{(\ln n)^n}} = \lim_{n\to\infty} \frac{e}{\ln(n+1)} \left(\frac{\ln n}{\ln(n+1)}\right)^n .$$
Because $\ln x$ is a strictly increasing function, $\ln n < \ln(n+1)$ and thus $\frac{\ln n}{\ln(n+1)} < 1$. It follows that
$$0 \le \lim_{n\to\infty} \frac{e}{\ln(n+1)} \left(\frac{\ln n}{\ln n+1}\right)^n \le \lim_{n\to\infty} \frac{e}{\ln(n+1)} = 0\,.$$
Therefore, $\lim_{n\to\infty} \frac{e}{\ln(n+1)} \left(\frac{\ln n}{\ln n+1}\right)^n = 0 < 1$ and the series $\sum \frac{e^n}{(\ln n)^n}$ converges.

18. Since $|\sin(n\pi/7)| \le 1$, we have
$$\left|\frac{\sin(n\pi/7)}{n^3}\right| \le \frac{1}{n^3}\,.$$

Because the series $\sum n^{-3}$ converges (it's a p-series and $p = 3 > 1$), the comparison test implies that the series $\sum \left|\frac{\sin(n\pi/7)}{n^3}\right|$ converges. Finally, the absolute convergence test implies $\sum \frac{\sin(n\pi/7)}{n^3}$ converges.

19. Applying the ratio test, we have

$$\lim_{n\to\infty} \frac{\left|\frac{(-2)^{n+1}}{(n+1)!}\right|}{\left|\frac{(-2)^n}{n!}\right|} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1\,,$$

and therefore the series $\sum \frac{(-2)^n}{n!}$ converges.

20. For all positive integers n, we have

$$\left(1 + \frac{1}{n}\right)^n \geq 1\,.$$

It follows that $\lim_{n\to\infty}(-1)^n\left(1+\frac{1}{n}\right)^n \neq 0$ and hence the series $\sum(-1)^n\left(1+\frac{1}{n}\right)^n$ diverges.

21. Applying the ratio test, we have

$$\lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}}{(2n+3)!}\right|}{\left|\frac{(-1)^n}{(2n+1)!}\right|} = \lim_{n\to\infty} \frac{(2n+1)!}{(2n+3)!} = \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} = 0 < 1\,.$$

Therefore, the series $\sum \frac{(-1)^n}{(2n+1)!}$ converges.

22. Using L'Hôpital's rule, we have

$$\lim_{n\to\infty} \frac{\ln n}{\ln(\ln n)} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n\ln n}} = \lim_{n\to\infty} \ln n = \infty\,.$$

Therefore, the series $\sum \frac{\ln n}{\ln(\ln n)}$ diverges.

23. Since $\sqrt{x}$ is a strictly increasing function, the inequality $n^3 < n(n+1)(n+2)$ implies $n^{3/2} < \sqrt{n(n+1)(n+2)}$ and

$$0 < \frac{1}{\sqrt{n(n+1)(n+2)}} < n^{-3/2}\,.$$

Because $\sum n^{-3/2}$ converges (it's a p-series with $p = 3/2 > 1$), the comparison test implies that the series $\sum \frac{1}{\sqrt{n(n+1)(n+2)}}$ also converges.

24. For $n \geq 2$, the function $f(n) = \frac{1}{2}n^2 - 1$ is positive; $f'(n) = n$ is positive and $f(2) = 1$. Adding $\frac{1}{2}n^2$ to both sides of the inequality $0 < \frac{1}{2}n^2 - 1$ yields $\frac{1}{2}n^2 < n^2 - 1$ for $n \geq 2$. Since the square root function is strictly increasing, we have $\frac{1}{\sqrt{2}}n < \sqrt{n^2-1}$ and

$$\frac{1}{n\sqrt{n^2-1}} < \frac{1}{\sqrt{2}n^2}$$

for $n \geq 3$. Since $\sum \frac{1}{\sqrt{2}n^2}$ converges (it's a p-series with $p = 2 > 1$), the comparison test implies that $\sum \frac{1}{n\sqrt{n^2-1}}$ also converges.

25. Using L'Hôpital's rule, we have

$$\lim_{n\to\infty} \frac{3\sqrt{n+1}}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{\frac{3}{2\sqrt{n+1}}}{\frac{1}{2\sqrt{n}}} = 3\sqrt{\lim_{n\to\infty} \frac{n}{n+1}} = 3 \neq 0\,.$$

Therefore, $\lim_{n\to\infty}(-1)^{n+1}\frac{3\sqrt{n+1}}{\sqrt{n}+1} \neq 0$ which implies that the series $\sum(-1)^{n+1}\frac{3\sqrt{n+1}}{\sqrt{n}+1}$ diverges.

26. If $f(n) = \ln(1 + 1/n)$ then

$$f'(n) = \left(\frac{1}{1+\frac{1}{n}}\right)\left(-\frac{1}{n^2}\right) = -\left(\frac{1}{n^2+n}\right)\,.$$

When $n > 0$, $f'(n) < 0$ and therefore $f(n)$ is decreasing. Moreover,

$$\lim_{n\to\infty} \ln\left(1 + \frac{1}{n}\right) = \ln 1 = 0\,,$$

and hence the alternating series test implies $\sum(-1)^n \ln\left(1 + \frac{1}{n}\right)$ converges.

27. If n is a positive integer then $\cos(n\pi) = (-1)^n$. Clearly

$$\frac{1}{n+1} < \frac{1}{n} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} = 0\,.$$

Therefore, the alternating series test implies that the series $\sum \frac{\cos n\pi}{n}$ converges.

28. For $n \geq 3$, the function $f(n) = \frac{1}{2}n^3 - 5$ is positive; $f'(n) = \frac{3}{2}n^2$ is positive and $f(3) = \frac{27}{2} - 5 > 0$. Adding $\frac{1}{2}n^3$ to both sides of the inequality $0 < \frac{1}{2}n^3 - 5$ yields $\frac{1}{2}n^3 < n^3 - 5$ which implies

$$\frac{1}{n^3-5} < \frac{2}{n^3}$$

for $n \geq 3$. Since $\sum 2n^{-3}$ converges (it's a p-series with $p = 3 > 1$), the comparison test implies that $\sum \frac{1}{n^3-5}$ also converges.

29. If $f(n) = (n^2 + 2n + 1)^{-1} = (n+1)^{-2}$ then $f'(n) = -2(n+1)^{-3}$. When $n > 0$, $f'(n) < 0$ and $f(n)$ is decreasing. Moreover,

$$\lim_{n\to\infty} \frac{1}{n^2+2n+1} = 0\,,$$

and therefore the alternating series test implies $\sum \frac{(-1)^{n-1}}{n^2+2n+1}$ converges.

30. For all $n \geq 1$, we have

$$\frac{(2n)!}{2^n \cdot n! \cdot n} = \frac{(2n)\cdot(2n-1)\cdot\cdots\cdot(n+1)}{2^n n} = \left(\frac{2n}{2n}\right)\left(\frac{2n-1}{2}\right)\cdots\left(\frac{n+1}{2}\right) \geq 1\,.$$

Hence $\lim_{n\to\infty}(-1)^n\frac{(2n)!}{2^n\cdot n!\cdot n} \neq 0$ and the series $\sum(-1)^n\frac{(2n)!}{2^n\cdot n!\cdot n}$ diverges.

31. Since $5^n < 5^n + n$, we have

$$\frac{1}{n+5^n} < \frac{1}{5^n} \text{ which implies } \frac{2^{n+1}}{n+5^n} < \frac{2^{n+1}}{5^n} = 2\left(\frac{2}{5}\right)^n .$$

Because $2/5 < 1$ the geometric series $\sum 2\left(\frac{2}{5}\right)^n$ converges. Applying the comparison test, we conclude that $\sum \frac{2^{n+1}}{n+5^n}$ also converges. Finally, $|(-2)^{n+1}| = 2^{n+1}$ so the absolute convergence test implies that the series $\sum \frac{(-2)^{n+1}}{n+5^n}$ converges.

32. Since $|\sin n| < 1$, we have

$$\left|\frac{(-1)^n \sin n}{n^2}\right| \le \frac{1}{n^2} .$$

Since the series $\sum n^{-2}$ converges (it's a p-series with $p = 2 > 1$), the comparison test implies that the series $\sum \frac{\sin n}{n^2}$ also converges. Finally, the absolute convergence test implies that $\sum (-1)^n \frac{\sin n}{n^2}$ converges.

33. Since $e^n < e^n + 1$, we have

$$\frac{2}{e^n+1} < \frac{2}{e^n} .$$

Since the series $2\sum e^{-n}$ converges (it's a geometric series and $e^{-1} < 1$), the comparison test implies that the series $\sum \frac{2}{e^n+1}$ also converges.

34. Since

$$\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\left(-\frac{1}{n^2}\right)\cos\left(\frac{1}{n}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos 0 = 1 \neq 0 ,$$

the series $\sum n \sin\left(\frac{1}{n}\right)$ diverges.

35. Because $f(x) = \frac{1}{x(\ln x)^p}$ is positive, continuous and decreasing on $[2, \infty)$, we may apply the integral test. Since $p > 1$, we have

$$\begin{aligned}
\int_2^\infty \frac{1}{x(\ln x)^p}\,dx &= \lim_{b\to\infty}\left[\frac{1}{1-p}(\ln n)^{1-p}\right]_0^b \\
&= \frac{1}{p-1}(\ln 2)^{1-p} + \lim_{b\to\infty} \frac{1}{1-p}(\ln b)^{1-p} \\
&= \frac{1}{(p-1)(\ln 2)^{p-1}} ,
\end{aligned}$$

and therefore the series $\sum \frac{1}{n(\ln n)^p}$ converges.

36. If $f(n) = (\sqrt{n} + \sqrt{n+1})^{-1}$ then

$$f'(n) = -(\sqrt{n} + \sqrt{n+1})^{-2}\left(\frac{1}{2\sqrt{n}} + \frac{1}{2\sqrt{n+1}}\right) .$$

When $n > 0$, $f'(n) < 0$ and $f(n)$ is decreasing. Moreover,

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}} = 0\,,$$

so the alternating series test implies $\sum \frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$ converges.