Lagrange Form of the Remainder (also called Lagrange Error Bound or Taylor's Theorem Remainder)

Given: $f(x) =$ power series in x

A **partial sum** is the sum of the first "few" terms of the series.
The **tail** is the rest of the terms of the series after a partial sum.
the **remainder** is the number you get by "adding" all the terms in the tail.
So $f(x) =$ partial sum + remainder

The **error** is the error you make by assuming $f(x) =$ the partial sum.

So the **error** is the same number as the **remainder** (obvious, but subtle)
An **error bound** is a number known to be greater than the absolute value of the remainder.

For an **alternating series** that converges (signs of the terms alternate, terms decrease in absolute value, limit of the absolute value of the terms goes to zero), the absolute value of the first term of the tail is an error bound.

In the **integral test for convergence**, the improper integral is an error bound.

Now, consider what Monsieur Lagrange is credited with showing. The **LAGRANGE REMAINDER** (the error) is exactly equal to the first term of the tail, but with its derivative evaluated not at $x = c$ (about which the series is expanded) but at some number z which is between c and the value of x at which you are evaluating the function. As this value of z comes from (repeated) application of the mean value theorem, there is often no way of knowing exactly what z equals. But if you can find a number that is an upper bound for the derivative between c and x, then you can find a **LAGRANGE ERROR BOUND**.

Taylor's Theorem: If a function f is differentiable through order $n+1$ in an interval containing c, then for each x in the interval, there exists a number z between x and c such that

$$f(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\ldots+\frac{f^{(n)}(c)}{n!}(x-c)^n+R_n(x)$$

$$\text{where } R_n(x)=\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

One useful consequence of Taylor's Theorem is that

$$R_n(x)=\frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|x-c\right|^{n+1},$$

where $\max\left|f^{(n+1)}(z)\right|$ is the maximum value of $f^{(n+1)}(z)$ where z is between x and c.

This gives us a **bound** for the error. It does not give us the exact value of the error. The bound is called **Lagrange's form of the remainder** or the **Lagrange error bound**.

The goal is to maximize $f^{(n+1)}(z)$ or to find a z that will maximize it. So – how do we do this? Here are the possibilities:

1. They tell you the maximum of $f^{(n+1)}(z)$ for some z between x and c. (See 2004B BC2 and 1999 BC4)
2. They give you a graph of $f^{(n+1)}(x)$ and you find its upper bound between x and c. (See 2011 BC6)
3. $f^{(n+1)}(x)$ is $\sin x$ or $\cos x$ in which case its maximum is 1. (See 2004 BC6)
4. $f^{(n+1)}(x)$ is increasing between x and c so its maximum is at the right endpoint (if it is decreasing then its maximum is at its left endpoint.) (See 2008 BC3)
5. It is a calculator problem so you graph $f^{(n+1)}(x)$ and determine its maximum between x and c.

LAGRANGE REMAINDER OR ERROR BOUND

Like alternating series, there is a way to tell how accurately your Taylor polynomial approximates the actual function value: you use something called the **Lagrange remainder** or **Lagrange error bound.**

Lagrange Remainder: If you use a Taylor polynomial of degree n centered about c to approximate the value x, then the actual function value falls within the error bound

$$\boxed{R_n(x) = \frac{f^{(n+1)}(z)\,(x-c)^{n+1}}{(n+1)!}\quad \text{, where z is some number between x and c.}}$$

Translation: Similar to alternating series, the error bound is given by the next term in the series, n + 1. the only tricky part is that you evaluate $f^{(n+1)}$, the (n + 1)th derivative, at z, not c. z is the number that makes $f^{(n+1)}(z)$ as large as it can be. This error bound is supposed to tell you how far off you are from the real number, so we want to assume the worst. We want the error bound to represent the largest possible error. In practice, picking z is pretty easy.

Example 1:

Approximate $\cos(.1)$ using a fourth-degree Maclaurin polynomial, and find the associated Lagrange remainder (error bound).

Solution:

Since the 4th degree Taylor polynomial for $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$, then

$$\cos(.1) \approx 1 - \frac{(.1)^2}{2!} + \frac{(.1)^4}{4!} \approx \boxed{.99500416667}$$

Now, the associated Lagrange remainder after n = 4 (denoted $R_4(x)$) is

$$R_4(x) = \frac{f^{(5)}(z)\,(x-c)^5}{5!}\,.$$

The fifth derivative of cos x is –sin z. Now, plug in x = .1 and c = 0 to get

$$R_4(.1) = \frac{(-\sin z)(.1)^5}{5!}\,.$$

We need – sin z to be as large as possible. The largest value of – sin z is 1. By assuming – sin z is the largest possible value, we are creating the largest possible error; so, plug in 1 for – sin z. The actual remainder will be less that this largest possible value.

$$R_4(.1) < \frac{(1)(.1)^5}{5!} = \frac{.1^5}{5!} = \boxed{.0000000833}$$

Therefore, our approximation of .99500416667 is off by less than .0000000833.

AP Calculus Workshop – Howard Alcosser

Example 2:

(a) Determine the degree of the Maclaurin polynomial that should be used to approximate $\sqrt[3]{e}$ to four decimal places. (b) Use this Maclaurin polynomial to estimate $\sqrt[3]{e}$ to four decimal places.

Solution:

(a) $f(x) = e^x$

The n^{th} degree Maclaurin polynomial is $P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!}$

The Lagrange form of the remainder with $x = \frac{1}{3}$ $\left(\text{since } \sqrt[3]{e} = e^{1/3}\right)$ is

$$R_n\!\left(\frac{1}{3}\right) = \frac{f^{(n+1)}(z)\left(\frac{1}{3}\right)^{n+1}}{(n+1)!} \quad \text{where } 0 < z < \frac{1}{3}$$

Since $f^{(n)}(x) = e^x$ for all derivatives of $f(x) = e^x$, we have

$$\left|R_n\!\left(\frac{1}{3}\right)\right| < \frac{e^{1/3}}{(n+1)!}\left(\frac{1}{3}\right)^{n+1} \quad \text{So, } \left|R_n\!\left(\frac{1}{3}\right)\right| < \frac{e^{1/3}}{(n+1)!\,3^{n+1}}$$

but since $e < 27$, then $e^{1/3} < 3$ and we have:

$$\left|R_n\!\left(\frac{1}{3}\right)\right| < \frac{3}{(n+1)!\,3^{n+1}} \quad \text{hence, } \left|R_n\!\left(\frac{1}{3}\right)\right| < \frac{1}{(n+1)!\,3^n} \quad \text{(the Lagrange error bound)}$$

Since we are seeking $\sqrt[3]{e}$ with four decimal accuracy, we need $\left|R_n\left(\frac{1}{3}\right)\right|$ to be less than 0.00005. So, $\left|R_n\!\left(\frac{1}{3}\right)\right| < 0.00005$ when $\frac{1}{(n+1)!\,3^n} < 0.00005$

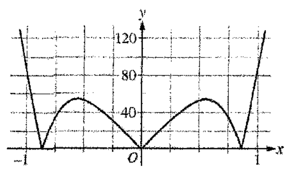
By trial and error using a calculator, this is true when n = 5 since $\frac{1}{(5+1)!\,3^5} \approx .000006 < .00005$

Therefore, we use $\boxed{P_5\left(\frac{1}{3}\right)}$ as an approximated value of $\sqrt[3]{e}$ accurate to 4 decimal places.

(b) Then $P_5(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$

$$\text{So}\quad P_5\!\left(\frac{1}{3}\right) = 1 + \left(\frac{1}{3}\right) + \frac{(1/3)^2}{2!} + \frac{(1/3)^3}{3!} + \frac{(1/3)^4}{4!} + \frac{(1/3)^5}{5!} = \frac{5087}{3645} \approx 1.39561$$

2011 AP® CALCULUS BC FREE-RESPONSE QUESTIONS



Graph of $y = \left|f^{(5)}(x)\right|$

6. Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

2008 AP® CALCULUS BC FREE-RESPONSE QUESTIONS

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

3. Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

(a) Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.

(b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

(c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4}.

2004 AP® CALCULUS BC FREE-RESPONSE QUESTIONS

6. Let f be the function given by $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$.

(a) Find $P(x)$.

(b) Find the coefficient of x^{22} in the Taylor series for f about $x = 0$.

(c) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right)\right| < \frac{1}{100}$.

(d) Let G be the function given by $G(x) = \int_0^x f(t)\,dt$. Write the third-degree Taylor polynomial for G about $x = 0$.

2004 AP CALCULUS BC FREE-RESPONSE QUESTIONS (Form B)

2.

Let f be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x = 2$ is given by $T(x) = 7 - 9(x-2)^2 - 3(x-2)^3$.

(a) Find $f(2)$ and $f''(2)$.

(b) Is there enough information given to determine whether f has a critical point at $x = 2$? If not, explain why not. If so, determine whether $f(2)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(c) Use $T(x)$ to find an approximation for $f(0)$. Is there enough information given to determine whether f has a critical point at $x = 0$? If not, explain why not. If so, determine whether $f(0)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(d) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 6$ for all x in the closed interval $[0, 2]$. Use the Lagrange error bound on the approximation to $f(0)$ found in part (c) to explain why $f(0)$ is negative.

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4. The function f has derivatives of all orders for all real numbers x. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.

(a) Write the third-degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.

(b) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 3$ for all x in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \ne -5$.

(c) Write the fourth-degree Taylor polynomial, $P(x)$, for $g(x) = f\left(x^2 + 2\right)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

AP® CALCULUS BC
2011 SCORING GUIDELINES

Question 6

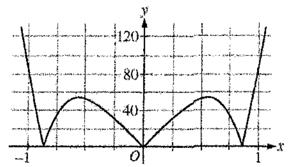
Let $f(x) = \sin(x^2) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin(x^2)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin(x^2)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.



Graph of $y = \left|f^{(5)}(x)\right|$

(a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ $\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$	$3: \begin{cases} 1: \text{series for } \sin x \\ 2: \text{series for } \sin(x^2) \end{cases}$
(b) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ $f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{121x^6}{6!} + \cdots$	$3: \begin{cases} 1: \text{series for } \cos x \\ 2: \text{series for } f(x) \end{cases}$
(c) $\frac{f^{(6)}(0)}{6!}$ is the coefficient of x^6 in the Taylor series for f about $x = 0$. Therefore $f^{(6)}(0) = -121$.	1 : answer
(d) The graph of $y = \left|f^{(5)}(x)\right|$ indicates that $\max\limits_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right| < 40$. Therefore $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| \le \frac{\max\limits_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right|}{5!} \cdot \left(\frac{1}{4}\right)^5 < \frac{40}{120 \cdot 4^5} = \frac{1}{3072} < \frac{1}{3000}$.	$2: \begin{cases} 1: \text{form of the error bound} \\ 1: \text{analysis} \end{cases}$

AP® CALCULUS BC
2008 SCORING GUIDELINES

Question 3

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

(a) Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.

(b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

(c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4}.

(a) $P_1(x) = 80 + 128(x-2)$, so $h(1.9) \approx P_1(1.9) = 67.2$ $P_1(1.9) < h(1.9)$ since h' is increasing on the interval $1 \le x \le 3$.	$4: \begin{cases} 2: P_1(x) \\ 1: P_1(1.9) \\ 1: P_1(1.9) < h(1.9) \text{ with reason} \end{cases}$
(b) $P_3(x) = 80 + 128(x-2) + \frac{488}{6}(x-2)^2 + \frac{448}{18}(x-2)^3$ $h(1.9) \approx P_3(1.9) = 67.988$	$3: \begin{cases} 2: P_3(x) \\ 1: P_3(1.9) \end{cases}$
(c) The fourth derivative of h is increasing on the interval $1 \le x \le 3$, so $\max\limits_{1.9 \le x \le 2} \left|h^{(4)}(x)\right| = \frac{584}{9}$. Therefore, $\left|h(1.9) - P_3(1.9)\right| \le \frac{584}{9} \frac{\left|1.9 - 2\right|^4}{4!}$ $= 2.7037 \times 10^{-4}$ $< 3 \times 10^{-4}$	$2: \begin{cases} 1: \text{form of Lagrange error estimate} \\ 1: \text{reasoning} \end{cases}$

AP® CALCULUS BC
2004 SCORING GUIDELINES

Question 6

Let f be the function given by $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$.

(a) Find $P(x)$.

(b) Find the coefficient of x^{22} in the Taylor series for f about $x = 0$.

(c) Use the Lagrange error bound to show that $\left| f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right) \right| < \frac{1}{100}$.

(d) Let G be the function given by $G(x) = \int_0^x f(t)\,dt$. Write the third-degree Taylor polynomial for G about $x = 0$.

(a) $f(0) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ $f'(0) = 5\cos\left(\frac{\pi}{4}\right) = \frac{5\sqrt{2}}{2}$ $f''(0) = -25\sin\left(\frac{\pi}{4}\right) = -\frac{25\sqrt{2}}{2}$ $f'''(0) = -125\cos\left(\frac{\pi}{4}\right) = -\frac{125\sqrt{2}}{2}$ $P(x) = \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}x - \frac{25\sqrt{2}}{2(2!)}x^2 - \frac{125\sqrt{2}}{2(3!)}x^3$	4 : $P(x)$ $\langle -1 \rangle$ each error or missing term deduct only once for $\sin\left(\frac{\pi}{4}\right)$ evaluation error deduct only once for $\cos\left(\frac{\pi}{4}\right)$ evaluation error $\langle -1 \rangle$ max for all extra terms, $+\cdots$, misuse of equality
(b) $\frac{-5^{22}\sqrt{2}}{2(22!)}$	2 : $\begin{cases} 1 : \text{magnitude} \\ 1 : \text{sign} \end{cases}$
(c) $\left| f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right) \right| \le \max\limits_{0 \le c \le \frac{1}{10}} \left| f^{(4)}(c) \right| \left(\frac{1}{4!}\right)\left(\frac{1}{10}\right)^4$ $\le \frac{625}{4!}\left(\frac{1}{10}\right)^4 = \frac{1}{384} < \frac{1}{100}$	1 : error bound in an appropriate inequality
(d) The third-degree Taylor polynomial for G about $x = 0$ is $\int_0^x \left(\frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}t - \frac{25\sqrt{2}}{4}t^2 \right) dt$ $= \frac{\sqrt{2}}{2}x + \frac{5\sqrt{2}}{4}x^2 - \frac{25\sqrt{2}}{12}x^3$	2 : third-degree Taylor polynomial for G about $x = 0$ $\langle -1 \rangle$ each incorrect or missing term $\langle -1 \rangle$ max for all extra terms, $+\cdots$, misuse of equality

AP® CALCULUS BC
2004 SCORING GUIDELINES (Form B)

Question 2

Let f be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x = 2$ is given by $T(x) = 7 - 9(x-2)^2 - 3(x-2)^3$.

(a) Find $f(2)$ and $f''(2)$.

(b) Is there enough information given to determine whether f has a critical point at $x = 2$? If not, explain why not. If so, determine whether $f(2)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(c) Use $T(x)$ to find an approximation for $f(0)$. Is there enough information given to determine whether f has a critical point at $x = 0$? If not, explain why not. If so, determine whether $f(0)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(d) The fourth derivative of f satisfies the inequality $\left| f^{(4)}(x) \right| \le 6$ for all x in the closed interval $[0, 2]$. Use the Lagrange error bound on the approximation to $f(0)$ found in part (c) to explain why $f(0)$ is negative.

(a) $f(2) = T(2) = 7$ $\frac{f''(2)}{2!} = -9$ so $f''(2) = -18$	2 : $\begin{cases} 1 : f(2) = 7 \\ 1 : f''(2) = -18 \end{cases}$
(b) Yes, since $f'(2) = T'(2) = 0$, f does have a critical point at $x = 2$. Since $f''(2) = -18 < 0$, $f(2)$ is a relative maximum value.	2 : $\begin{cases} 1 : \text{states } f'(2) = 0 \\ 1 : \text{declares } f(2) \text{ as a relative maximum because } f''(2) < 0 \end{cases}$
(c) $f(0) \approx T(0) = -5$ It is not possible to determine if f has a critical point at $x = 0$ because $T(x)$ gives exact information only at $x = 2$.	3 : $\begin{cases} 1 : f(0) \approx T(0) = -5 \\ 1 : \text{declares that it is not possible to determine} \\ 1 : \text{reason} \end{cases}$
(d) Lagrange error bound $= \frac{6}{4!}\left|0 - 2\right|^4 = 4$ $f(0) \le T(0) + 4 = -1$ Therefore, $f(0)$ is negative.	2 : $\begin{cases} 1 : \text{value of Lagrange error bound} \\ 1 : \text{explanation} \end{cases}$

BC–4 **1999**

4. The function f has derivatives of all orders for all real numbers x. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.

   (a) Write the third-degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.

   (b) The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \le 3$ for all x in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \ne -5$.

   (c) Write the fourth-degree Taylor polynomial, $P(x)$, for $g(x) = f(x^2 + 2)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

(a) $T_3(f,2)(x) = -3 + 5(x-2) + \frac{3}{2}(x-2)^2 - \frac{8}{6}(x-2)^3$

$f(1.5) \approx T_3(f,2)(1.5)$

$= -3 + 5(-0.5) + \frac{3}{2}(-0.5)^2 - \frac{4}{3}(-0.5)^3$

$= -4.958\overline{3} = -4.958$

4 $\begin{cases} 3\text{: } T_3(f,2)(x) \\ \quad <-1> \text{ each error} \\ 1\text{: approximation of } f(1.5) \end{cases}$

(b) Lagrange Error Bound $= \frac{3}{4!}|1.5-2|^4 = 0.0078125$

$f(1.5) > -4.958\overline{3} - 0.0078125 = -4.966 > -5$

Therefore, $f(1.5) \ne -5$.

2 $\begin{cases} 1\text{: value of Lagrange Error Bound} \\ 1\text{: explanation} \end{cases}$

(c) $P(x) = T_4(g,0)(x)$

$= T_2(f,2)(x^2+2) = -3 + 5x^2 + \frac{3}{2}x^4$

The coefficient of x in $P(x)$ is $g'(0)$. This coefficient is 0, so $g'(0) = 0$.

The coefficient of x^2 in $P(x)$ is $\frac{g''(0)}{2!}$. This coefficient is 5, so $g''(0) = 10$ which is greater than 0.

Therefore, g has a relative minimum at $x = 0$.

3 $\begin{cases} 2\text{: } T_4(g,0)(x) \\ \quad <-1> \text{ each incorrect, missing, or extra term} \\ 1\text{: explanation} \end{cases}$

Note:
$<-1>$ max for improper use of $+\ldots$ or equality