6-3 Worksheet

Problems

1. Solve the differential equation $\frac{dy}{dx} = \sqrt{xy}$.

2. Solve the differential equation $y \ln x - xy' = 0$.

3. Find the particular solution of the differential equation $\frac{du}{dv} = uv \sin v^2$, $u(0) = 1$.

4. Find the orthogonal trajectories of the family $y = Ce^x$.

5. Find the orthogonal trajectories of the family $x^2 = Cy$.

6. Verify that $y = \frac{L}{1+be^{-kt}}$ satisfies the logistic differential equation $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$.

7. Solve the logistic differential equation $\frac{dy}{dt} = y\left(1 - \frac{y}{36}\right)$, $y(0) = 4$.

8. Solve the logistic differential equation $\frac{dy}{dt} = \frac{4y}{5} - \frac{y^2}{150}$, $y(0) = 8$.

9. At time $t = 0$, a bacterial culture weighs 1 gram. Two hours later, the culture weighs 4 grams. The maximum weight of the culture is 20 grams. Write a logistic equation that models the weight of the bacterial culture. Then, use your model to find the weight after 5 hours.

10. Write the differential equation that models the following verbal statement: The rate of change of y with respect to x is proportional to the difference between y and 4.

Solutions

1. $\frac{dy}{y} = \sqrt{x}\,dx \Rightarrow \int \frac{dy}{y} = \int \sqrt{x}\,dx$. So, we have $\ln|y| = \frac{2}{3}x^{3/2} + C_1 \Rightarrow y = e^{(2/3)x^{3/2}+C_1} = Ce^{(2/3)x^{3/2}}$.

2. $y\ln x = x\frac{dy}{dx} \Rightarrow \frac{\ln x}{x}dx = \frac{dy}{y} \Rightarrow \int \frac{\ln x}{x}dx = \int \frac{dy}{y}$. Hence, we have

   $\frac{1}{2}(\ln|x|)^2 + C_1 = \ln|y| \Rightarrow y = e^{(1/2)(\ln x)^2 + C_1} = Ce^{(\ln x)^2/2}$. Note that $x > 0$.

3. $\int \frac{du}{u} = \int v\sin v^2\,dv \Rightarrow \ln|u| = -\frac{1}{2}\cos v^2 + C_1 \Rightarrow u = Ce^{-(\cos v^2)/2}$. The initial condition gives

   $u(0) = 1 = Ce^{-1/2} \Rightarrow C = e^{1/2}$. The particular solution is, therefore,

   $u = e^{1/2}e^{-(\cos v^2)/2} = e^{(1-\cos v^2)/2}$.

4. The given family of exponential functions is $y = Ce^x \Rightarrow y' = Ce^x = y$. The orthogonal trajectories satisfy

   $y' = -\frac{1}{y} \Rightarrow \frac{dy}{dx} = -\frac{1}{y} \Rightarrow \int y\,dy = -\int dx \Rightarrow \frac{y^2}{2} = -x + K_1$. Hence, the orthogonal trajectories are the family of parabolas $y^2 = -2x + K$.

5. The given family of parabolas is $x^2 = Cy \Rightarrow 2x = Cy'$, and we can solve for y',

   as follows: $y' = \frac{2x}{C} = \frac{2x}{x^2/y} = \frac{2y}{x}$. The orthogonal trajectories satisfy the equation

   $\frac{dy}{dx} = -\frac{x}{2y} \Rightarrow 2\int y\,dy = -\int dx \Rightarrow y^2 = -\frac{x^2}{2} + K_1$. Hence, the orthogonal trajectories are the family of ellipses $x^2 + 2y^2 = K$.

6. $y = \frac{L}{1+be^{-kt}} = L\left(1+be^{-kt}\right)^{-1} \Rightarrow y' = -L\left(1+be^{-kt}\right)^{-2}\left(-kbe^{-kt}\right)$. This can be rearranged as follows:

$$
\begin{aligned}
y' &= -L\left(1+be^{-kt}\right)^{-2}\left(-kbe^{-kt}\right) \\
&= k\left[\frac{L}{1+be^{-kt}}\right]\left[\frac{1}{1+be^{-kt}}\left(be^{-kt}\right)\right] \\
&= ky\left[\frac{1+be^{-kt}-1}{1+be^{-kt}}\right] \\
&= ky\left[1-\frac{1}{1+be^{-kt}}\right] \\
&= ky\left[1-\frac{L}{L\left(1+be^{-kt}\right)}\right] = ky\left(1-\frac{y}{L}\right).
\end{aligned}
$$

7. For this equation, $k=1$ and $L=36$. Therefore, the solution is $y=\dfrac{L}{1+be^{-kt}}=\dfrac{36}{1+be^{-t}}$. We determine $\wp$ by using the initial condition:

$4=\dfrac{36}{1+b}\Rightarrow b=8$. Hence, the solution is $y=\dfrac{36}{1+8e^{-t}}$.

8. We can rewrite the equation as $\dfrac{dy}{dt}=\dfrac{4}{5}y\left(1-\dfrac{y}{120}\right)$. For this equation, $k=\dfrac{4}{5}=0.8$ and $L=120$. Therefore, the solution is $y=\dfrac{L}{1+be^{-kt}}=\dfrac{120}{1+be^{-0.8t}}$.

We determine b by using the initial condition:

$8=\dfrac{120}{1+b}\Rightarrow b=14$. Hence, the solution is $y=\dfrac{120}{1+14e^{-0.8t}}$.

9. The model is $y=\dfrac{L}{1+be^{-kt}}$, $L=20$, $y(0)=1$, $y(2)=4$. Hence, $y=\dfrac{20}{1+be^{-kt}}$.

$y(0)=1\Rightarrow 1=\dfrac{20}{1+b}\Rightarrow b=19$. $y(2)=4\Rightarrow 4=\dfrac{20}{1+19e^{-2k}}$. Solving for k,

$1+19e^{-2k}=5\Rightarrow e^{-2k}=\dfrac{4}{19}\Rightarrow -2k=\ln\dfrac{4}{19}\Rightarrow k=\dfrac{1}{2}\ln\dfrac{19}{4}\approx 0.7791$. The logistic model is

$y=\dfrac{20}{1+19e^{-0.7791t}}$. At $t=5, y\approx 14.43$ grams.

10. $\dfrac{dy}{dx}=k\left(y-4\right)$.